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Eigenvalue correlations in the circular ensembles

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Abstract. Dyson introduced two types of Brownian-motion ensembles of random matrices for studying approximate symmetries in complex quantum systems. The magnitude of symmetry breaking plays the role of a fictitious time $t \geq 0$. We study here the eigenvalue correlations in the circular-type ensembles which serve as models for the evolution operators of quantum maps with chaotic classical limits. In two cases involving time-reversal symmetry breaking we evaluate explicitly the eigenangle-density correlation functions of all orders for all t and for all values of the matrix dimensionality N. The general case is described by a hierarchic set of relations among the correlation functions. As a function of t, the transition in the correlations is found to be rapid for large N, discontinuous for $N \to \infty$. As a function of a local parameter Λ, which measures the mean square symmetry-admixing matrix element in units of the local average spacing, the transition is found to be smooth. The same Λ-dependent results were found earlier for the Gaussian-type ensembles which serve as models for the Hamiltonian operators of autonomous chaotic systems. We show elsewhere by a semiclassical calculation for a class of quantum maps with time-reversal breaking that the long-range correlations are identical to those obtained in this paper. Our results thus indicate a universality associated with the 'non-equilibrium' eigenvalue statistics.

1. Introduction

Recent studies on quantum chaos indicate that the distribution of eigenvalues (specifically, the local correlations and fluctuations in eigenvalue sequences) of complicated operators fall into universality classes which can be modelled by random matrices. Two types of random matrices have proved particularly useful. These are the Gaussian ensembles (GE) of Hermitian matrices and the circular ensembles (CE) of unitary matrices, serving respectively as models for the Hamiltonian operators of autonomous systems and for the evolution operators of quantum maps (i.e., systems with time-periodic Hamiltonians), the underlying classical dynamics in both cases being fully chaotic. For each of the two types, there are three important universality classes, determined by the invariance of the system under time-reversal (TR) transformation [1–4] (or more generally antiunitary transformations [5]) and described by the invariance of the ensemble measure: invariance under orthogonal (O) or symplectic (S) transformations for TR-invariant systems and under unitary (U) transformations for TR-non-invariant ones; the invariance restricts the allowed space of matrices, for example, to that of symmetric matrices for O-invariance. The GOE, GSE and GUE thus constitute the three universal GEs whereas the COE, CSE and CUE the three universal

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CEs. A semiclassical theory of the long-range correlations [6], numerical simulations [4, 7–11] of systems with few degrees of freedom, and experimental data on complex systems [12–15] confirm the validity of the universality.

The ensemble theory for the universality classes is akin to that in equilibrium statistical mechanics where no attention is paid to the approach to equilibrium. A non-equilibrium ensemble theory [16] would however be useful, for example, for an autonomous chaotic system with Hamiltonian H_α in which a symmetry is broken, α being a measure of the breaking. For TR symmetry, GOE statistics may apply for $\alpha=0$ but, at asymptotically large energies, GUE statistics would apply for any $\alpha\neq 0$; however, at intermediate energies with sufficiently small values of α an intermediate statistics would be obtained, indicative of a non-equilibrium behaviour. Similarly if the system is integrable with regular classical motion for $\alpha=0$, but fully chaotic for $\alpha\neq 0$, a Poisson$\rightarrow$GOE transition may be realized. Examples of such behaviour have been indicated in numerical simulations [8, 17, 18] as well as experimental data [13, 14].

For a time-periodic Hamiltonian, the evolution operator (for the period of the Hamiltonian) generates the quantum map. The map is defined in a finite (N)-dimensional Hilbert space, if the corresponding classical map is restricted to a finite phase space. For symmetry-breaking, involving α-dependent evolution operators U_α, the role of energy will be replaced by that of the dimension. For $N\rightarrow\infty$, the statistics would be that of one of the universality classes, but, for finite large N, intermediate statistics would again be observed for small α. Many such transitions (including COE$\rightarrow$CUE) can be studied numerically in the systems of kicked tops and rotators [4, 10, 11].

In order to model the approach to equilibrium, Dyson [19, 20] introduced Brownian-motion ensembles of random matrices, characterized by a fictitious 'time' t and a fictitious 'temperature' β^{-1}, which, for large t, yield the equilibrium ensembles (with $\beta=1, 4, 2$ respectively for O, S and U). When applied to real systems, the parameter t will be a measure of symmetry breaking, related to α as discussed ahead. He showed that, in such ensembles, the dynamics of eigenvalues (constrained to move on the real line in GE and on the unit circle in CE) is identical to that of a set of charged particles, moving under the mutual repulsion of a two-dimensional Coulomb (logarithmic) potential and executing Brownian motion in time t. He also conjectured that the transition to equilibrium will be rapid, discontinuous for infinite-dimensional matrices, as a function of t.

The Gaussian type Brownian ensembles have been the subject of several recent studies [16, 21–24]. They confirm the Dyson conjecture and show moreover that the transition in the energy-level correlation functions is governed, for small t and large matrix dimensionality N, by an energy-dependent parameter Λ which measures locally the mean-square symmetry breaking (off-diagonal) matrix element in units of the average level spacing; it is zero for $t=0$ and its large values mark the completion of the transition. For GOE$\rightarrow$GUE and GSE$\rightarrow$GUE transitions, the correlation functions of all orders and for all Λ have been explicitly evaluated [22, 23]; for the other transitions, the correlation functions are given implicitly by a hierarchic set of relations [16].

Our purpose in this paper is to study the correlation functions in the circular type Brownian ensembles. We shall show that the same parameter Λ governs the transition in these ensembles also, and shall moreover establish that the correlation functions of all orders under similar initial conditions, and for the same Λ and β, are identical to those obtained in the Gaussian cases. For $t\rightarrow\infty$ (equilibrium cases), the latter result had been proved by Dyson [25] and Mehta [26]; but the extension to the non-equilibrium ensembles is not obvious. This suggests that there is a universality associated with the non-equilibrium statistics also.

In real systems t and Λ can be usually inferred from the infinitesimal variations of H_α or U_α. When $H_{\alpha+\delta\alpha} = H_\alpha + \delta\alpha V$ or $U_{\alpha+\delta\alpha} = U_\alpha(1+\mathrm{i}\delta\alpha V)$, we may take $t = \alpha^2$, $\delta t = (\delta\alpha)^2$ and $\Lambda = \alpha^2 v^2/D^2$; here D is the mean spacing ($=2\pi/N$ for quantum maps) and βv^2 is the mean of $|V_{ij}|^2$ with V_{ij} the near-diagonal matrix element of V in H_α or U_α-diagonal representation. The assumption here is that V, written in H_α- or U_α-diagonal representation, would be a random matrix.

Our current interest in the circular ensembles stems from the fact that the numerical simulations are relatively easier for the quantum maps and therefore all aspects of the non-equilibrium ensembles may be verifiable. We mention, however, that the GOE → GUE transition results have been used to obtain bounds on TR-non-invariance in the nuclear interaction [16]. Morever, a parameter qualitatively similar to Λ has been conjectured [8] for a chaotic billiard in a magnetic field; a very recent numerical calculation [27] confirms the GOE → GUE transition for the same system. In a later paper we shall use the ideas of [6, 8] for quantum maps to obtain semiclassically the Λ-dependent long-range correlations for TR-non-invariance.

In section 2 we shall give a brief review of the Dyson theory for the circular type Brownian ensembles and then obtain explicitly the joint-probability density of the eigenangles for $\beta = 2$. In sections 3 and 4 we shall use the $\beta = 2$ solution to derive, respectively for COE → CUE and CSE → CUE transitions, the correlation functions of all orders for all N. Our proofs will run parallel to those in the Gaussian transitions [23] and will rely on the elegant techniques developed by Dyson and Mehta [3, 25, 26] for the equilibrium ensembles. Some of the details of the derivations are included in appendices 1–3 and some additional results in appendix 4. In section 5 we shall derive the general hierarchic relations which, for large N, will be the same as those for the Gaussian cases [16]. In the concluding section we shall give a brief summary of our semiclassical result mentioned above.

2. The circular-type Brownian ensembles

2.1. *Preliminaries*

We shall abbreviate the two types of Brownian ensembles as $\mathrm{CE}_\beta(t)$ and $\mathrm{GE}_\beta(t)$. The basic Brownian 'steps' are given in terms of the equilibrium ensembles $\mathrm{GE}_\beta(\infty)$ which we define in this subsection.

We first define a self-dual N-dimensional matrix B [28, 29]. To take account of the threefold classification, we write $B = \Sigma B_r e_r$ where $r = 0, 1, \ldots, \beta - 1$ and the e_r are the quaternion units (i.e., $e_0 = 1$, $e_r e_{r'} + e_{r'} e_r = -2\delta_{rr'}$ for $r, r' \neq 0$). Then, for a self-dual B, B_r is symmetric for $r = 0$ and antisymmetric for $r \neq 0$, and, for a self-dual Hermitian B, the B_r are also real. To make correspondence with quantum problems, we replace e_r by their two-dimensional matrix representatives for $\beta = 4$, but, for $\beta = 1, 2$, we merely write $e_0 = 1$, $e_1 = i$.

The $\mathrm{GE}_\beta(\infty)$ is an ensemble [1–3] of self-dual Hermitian matrices M in which the distinct non-zero matrix elements of M_r are distributed independently as zero-centred Gaussian variables. The variances are v^2 for the off-diagonal matrix elements and $2v^2$ for the diagonal ones of M_0. Thus, with bar denoting ensemble averaging,

$$\overline{M_{r;ij}} = 0 \tag{1}$$

$$\overline{M_{r;ij} M_{r';kl}} = \delta_{rr'}(\delta_{ik}\delta_{jl} \pm \delta_{il}\delta_{jk}) v^2 \tag{2}$$

where in (2) the upper sign is for $r=0$ and the lower for $r\neq 0$. v^2 fixes the scale and, in the Brownian ensembles, can be absorbed in the definition of t. To begin with we choose $v^2=1$; a different choice [19] ($\beta f v^2=1$ where f is the 'friction coefficient') makes the correspondence with the Langevin equation in section 2.2 more precise; in section 5 we shall make other choices.

Equations (1) and (2) are representation-independent, as long as the symmetry of the system is preserved; for example, with $\beta=1$, all orthogonal transforms of M are contained in the GOE.

2.2. *Diffusion equation for* $\mathrm{CE}_\beta(t)$

The $\mathrm{CE}_\beta(t)$ is an ensemble of time-dependent self-dual unitary matrices $U(t)$ in which the symmetry-breaking is introduced at each time t as an infinitesimal random perturbation. For eigenvalue considerations, it is adequate to define it by

$$U(t+\delta t)=U(t)\exp(\mathrm{i}\sqrt{\delta t}\,M(t)) \tag{3}$$

where δt is infinitesimal and $M(t)$, independent for each t, is a member of $\mathrm{GE}_\beta(\infty)$. $U(0)$ is taken to be a diagonal unitary matrix.

Let $\theta_j(t)$ be the eigenangles of $U(t)$. Then, from the standard perturbation theory for unitary matrices,

$$\begin{aligned}\delta\theta_j &\equiv \theta_j(t+\delta t)-\theta_j(t)\\ &= M_{0:jj}(\delta t)^{1/2}+\tfrac{1}{2}\sum_{k(\neq j)}\sum_{r=0}^{\beta-1}(M_{r:jk})^2\cot\left(\frac{\theta_j-\theta_k}{2}\right)\delta t\end{aligned} \tag{4}$$

where in the last form, correct to first order in δt, the t-dependence of M and θ are understood. For fixed θ_j, we use (1), (2) in $U(t)$-diagonal representation to perform the averages. We find, to the same order in δt,

$$\overline{\delta\theta_j}=\beta E(\theta_j)\delta t \tag{5}$$

$$\overline{\delta\theta_j\delta\theta_k}=2\delta_{jk}\delta t \tag{6}$$

where

$$E(\theta_j)=\tfrac{1}{2}\sum_{k(\neq j)}\cot\left(\frac{\theta_j-\theta_k}{2}\right)=\frac{\partial}{\partial\theta_j}\log|Q_N| \tag{7}$$

$$Q_N(\theta)\equiv Q_N(\theta_1,\ldots,\theta_N)=\prod_{j<k}\sin\left(\frac{\theta_j-\theta_k}{2}\right) \tag{8}$$

with $Q_N=1$ for $N=1$. Equations (5)–(8) formally establish the equivalence of the Brownian motion of the eigenangles with that of a gas of unit charges on the unit circle under the mutually repulsive two-dimensional Coulomb forces [3, 19]. Note that the force $E(\theta_j)$ is the tangential component of the electric force felt by the charge at $\exp(\mathrm{i}\theta_j)$. The corresponding Langevin equation (representing an overdamped case) can be obtained from (4) by taking the $\delta t\to 0$ limit and replacing $(M_{r:jk})^2$, for $j\neq k$, by its average.

A diffusion equation for the joint-probability density of the eigenangles, $P(\theta; t)$, can be obtained from (5), (6):

$$\frac{\partial P}{\partial t} = \sum_j \frac{\partial^2 P}{\partial \theta_j^2} - \beta \sum_j \frac{\partial}{\partial \theta_j}(E(\theta_j)P)$$

$$= \sum_j \frac{\partial}{\partial \theta_j} |Q_N|^\beta \frac{\partial}{\partial \theta_j} \frac{P}{|Q_N|^\beta}. \tag{9}$$

Starting with an initial distribution (given by that of the eigenangles of $U(0)$) (9) gives P uniquely for all $t > 0$. The equilibrium distribution ($t \to \infty$) is obtained from the electric potential ($-\log|Q_n|$):

$$P(\theta; \infty) = C_\beta(N)|Q_N|^\beta \tag{10}$$

which can also be verified easily from the last form of (9) and is valid for COE, CUE and CSE depending on the value of β. See [3, 28] for the normalization constant $C_\beta(N)$. A similar diffusion equation is also obtained for $\mathrm{GE}_\beta(t)$ [16, 19]. Note that one can now extend (9) to other β values; $\beta = 0$ for example may be considered the analogue of a Poisson ensemble whereas $\beta = \infty$ corresponds to a uniform case with charges equally spaced on the circle.

2.3. *Formal solution of the diffusion equation*

The transformation $\xi = P/|Q_N|^{\beta/2}$ allows us to cast (9) in the suggestive form

$$\frac{\partial \xi}{\partial t} = -H\xi \tag{11}$$

where the 'Hamiltonian' H is given by

$$H = -\sum_j \frac{\partial^2}{\partial \theta_j^2} - \frac{\beta^2}{48} N(N^2-1) + \frac{\beta(\beta-2)}{16} \sum_{j \neq k} \operatorname{cosec}^2\left(\frac{\theta_j - \theta_k}{2}\right). \tag{12}$$

With periodic boundary conditions and the requirement (to take account of the singularity in H) that the solutions vanish as $|\theta_j - \theta_k|^{\beta/2}$ when θ_j and θ_k are close to each other, H has well-defined (completely symmetric) eigenstates ξ_k and eigenvalues λ_k. We have then

$$P(\theta, \phi; t) = \left|\frac{Q_N(\theta)}{Q_N(\phi)}\right|^{\beta/2} \sum_{k \geq 0} \exp(-\lambda_k t)\xi_k(\theta)\xi_k^*(\phi) \tag{13}$$

where $\phi \equiv (\phi_1, \ldots, \phi_N)$ are the eigenangles of $U(0)$. If a distribution were defined on ϕ also, then

$$P(\theta; t) = \int P(\theta, \phi; t)P(\phi; 0)\, \mathrm{d}\phi_1, \ldots, \mathrm{d}\phi_N. \tag{14}$$

Equations (13) and (14) represent the formal solution of (9). In keeping with (10), the ground state must be non-degenerate with $\lambda_0 = 0$, $\xi_0 = C_\beta^{1/2}|Q_N|^{\beta/2}$; all other eigenvalues must be positive. Sutherland [30] has discussed the Hamiltonian (12) and obtained the excited states; no compact solution for P has been obtained. However, for $\beta = 2$, the interaction term in (12) drops out and then, as discussed below, P can be obtained explicitly.

2.4. Solution for $\beta=2$

For $\beta=2$, it is slightly better to use the transformation $\tilde{\xi}=P/Q_N$. We again obtain (11) and (12) without the interaction term. Now $\tilde{\xi}$ must be completely antisymmetric since Q_N is. The eigenvalues and eigenvectors can therefore be written as

$$\tilde{\lambda}_k=\sum_{j=1}^{N} k_j^2-\frac{N(N^2-1)}{12} \tag{15}$$

$$\tilde{\xi}_k(\theta)=\left(\frac{1}{2\pi}\right)^{N/2}\left(\frac{1}{N!}\right)^{1/2}\det[\exp(\mathrm{i}k_j\theta_{j'})]_{j,j'=1,\ldots,N} \tag{16}$$

where the $\tilde{\xi}_k$ are appropriately antisymmetrized and $k\equiv(k_1,\ldots,k_N)$. The k_j are fixed by the periodic boundary conditions. When $\theta_j\to\theta_j+2\pi$ for any j, Q_N acquires a phase $(-1)^{N-1}$ and hence $\tilde{\xi}_k$ must also acquire the same phase. Thus $k_j=0,\pm1,\pm2,\ldots$, or, $\pm\frac{1}{2},\pm\frac{3}{2},\ldots$, depending on whether N is odd or even. Substitution of (15), (16) in (13) (with $|Q|\to Q$, $\lambda\to\tilde{\lambda}$ and $\xi\to\tilde{\xi}$) completes the solution. Note that the boundary condition for small separations is already built into (16).

A compact solution for P is obtained if we expand the determinants in terms of their matrix elements and then carry out the eigenvalue sum (Σ_k) first. The latter sum yields typically

$$\frac{1}{(2\pi)^N}\sum_{\{k_i\}}\prod_{j=1}^{N}\exp[-k_j^2t+\mathrm{i}k_j(\theta_{p_j}-\theta_{p'_j})]=\prod_{j=1}^{N}f(\theta_{p_j}-\phi_{p'_j};\,t) \tag{17}$$

where

$$f(\psi)=\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\exp[-k^2t+\mathrm{i}k\psi] \tag{18}$$

and p, p' denote independent permutations on the indices j. In (18) the values of k are integral if N is odd, half-integral if N is even. Now the determinantal sum $\Sigma_{p'}(-1)^{p'}$ gives a determinant while the other sum $\Sigma_p(-1)^p$ fixes the ordering of θ, ϕ indices and yields a factor $N!$ (these sums are equivalent to the antisymmetrization of (17) with respect to θ and ϕ both); the ordering of indices is the same as that in Q_N. The final result is

$$P(\theta,\phi;\,t;\,\beta=2)=\frac{1}{N!}\frac{Q_N(\theta)}{Q_N(\phi)}\exp\left(\frac{tN(N^2-1)}{12}\right)\det[f(\theta_i-\phi_j;\,t)]_{i,j=1\ldots N}. \tag{19}$$

To verify the equilibrium solution note that the ground state ($\lambda_0=0$) is given in (15), (16) by $k_j=(N-1)/2,\ (N-3)/2,\ldots,\ -(N-1)/2$. Since $\det[\exp(\mathrm{i}k_j\theta_{j'})]=2^{N(N-1)/2}Q_N(\theta)$, we obtain $P(\theta;\infty)=C_2(N)|Q_N|^2$ with $C_2(N)=(2\pi)^{-N}(N!)^{-1}2^{N(N-1)}$. A similar technique facilitates the solution of the $\mathrm{GE}_2(t)$ equation [23, 31].

2.5. Definition of correlation functions

The n-angle correlation function [25] is defined as

$$R_n(\theta_1,\ldots,\theta_n;\,t)=\frac{N!}{(N-n)!}\int \mathrm{d}\theta_{n+1}\ldots\int \mathrm{d}\theta_N\,P(\theta_1,\theta_2,\ldots,\theta_N;\,t) \tag{20}$$

where $R_N = N!\,P$. We would be interested in the large-N limit, for which the θ-spectrum must first be 'unfolded'. Using the unfolding function

$$r = \int_0^{\theta} R_1(\theta'; t)\,\mathrm{d}\theta' \tag{21}$$

we have the unfolded correlation function

$$\boldsymbol{R}_n(r_1, \ldots, r_n; \Lambda) = \lim_{N\to\infty} \frac{R_n(\theta_1 \ldots \theta_n; t)}{R_1(\theta_1; t) \ldots R_1(\theta_n; t)} \tag{22}$$

where the Λ-argument in $\boldsymbol{R}_n$ is included in anticipation of the results ahead. One can define cluster functions also which follow from the correlation functions [25]; see appendix 4. All fluctuation measures, *viz.* measures of eigenvalue regularity used in data analysis, follow from $\boldsymbol{R}_n$ [2, 16, 24]; we shall not deal with them here.

Starting with (19), we shall derive R_n and $\boldsymbol{R}_n$ in sections 3 and 4 respectively for the COE→CUE and CSE→CUE transitions. The hierarchic relations among $\boldsymbol{R}_n$ in section 5 will be derived from (9).

3. COE→CUE ensembles

In this section we shall obtain the correlation functions for the COE→CUE ensembles. The joint-probability density of the eigenangles is given by (14) and (19) with the initial distribution $P(\phi; 0) = C_1(N)|Q_N(\phi)|$ where $C_1(N) = (2\pi)^{-N}(\Gamma(\frac{3}{2}))^N\, 2^{N(N-1)/2}(\Gamma(1+N/2))^{-1}$. To work out the integral we shall use Mehta's [3] method of integration over alternate variables; the result will be obtained in terms of the Pfaffian (Pf) of an even-dimensional antisymmetric matrix A; $\mathrm{Pf}[A] \equiv (\det A)^{1/2}$ is expressible as a polynomial function of the matrix elements. For the correlation-function integrals we shall use Dyson's quaternion-determinant method [25, 29]; the quaternion determinant (Tdet) of a self-dual quaternion matrix B (section 2.1) is $(\det M(B))^{1/2}$ where $M(B)$ is the $(2N)$-dimensional matrix obtained from B by replacing the quaternions by their two-dimensional matrix representatives; $\mathrm{Tdet}(B)$ is also a (scalar) polynomial function of the quaternion matrix elements of B. Details of these methods can be found in the original references. We shall use equations (3.3)–(3.14) and (4.1)–(4.5) of [23] where similar integrals are encountered. We also define the Vandermonde determinant

$$\Delta_N(\theta) = \prod_{i<j} (\theta_i - \theta_j). \tag{23}$$

Note that $\mathrm{sign}(\Delta_N(\theta)) = \mathrm{sign}(Q_N(\theta))$.

3.1. *Joint-probability density for the eigenangles*

As explained above the joint-probability density in the present case can be written as

$$P(\theta; t) = \frac{C_1(N)}{N!} \exp\left(\frac{tN(N^2-1)}{12}\right) Q_N(\theta) \times \int_0^{2\pi} \ldots \int \mathrm{d}\phi_1 \ldots \mathrm{d}\phi_N\, \mathrm{sign}(\Delta_N(\phi)) \det[f(\theta_i - \phi_j; t)]. \tag{24}$$

Using (3.3), (3.6), (3.9) and (3.10)† of [23] we find (see appendix 1 for details):

$$P(\theta;\,t)=C_1(N)\exp\left(\frac{tN(N^2-1)}{12}\right)Q_N(\theta)\,\mathrm{Pf}[F_{ij}]. \tag{25}$$

Here F is a $2m$-dimensional antisymmetric matrix with $2m=N$ or $N+1$ as N is even or odd. The matrix elements F_{ij} are given, for $i, j=1,\ldots,N$, by

$$F_{ij}\equiv F(\theta_i-\theta_j\,;\,t)=\frac{2}{\pi}\sum_{k>0}\exp(-2k^2t)\,\frac{\sin k(\theta_i-\theta_j)}{k} \tag{26}$$

with k integer or half-integer as in (18). For odd N, we have in addition

$$F_{i,N+1}=-F_{N+1,i}=1 \tag{27}$$

for $i=1,\ldots,N$, and $F_{N+1,N+1}=0$. As in the Gaussian case [23], Pf in (25) contains the main t-dependence and, for large t, is proportional to $Q_N(\theta)$.

3.2. Correlation functions for finite N

Following closely the method of section 4 of [23] we write (25) in the form (see appendix 2 for proof)

$$P(\theta;\,t)=(N!)^{-1}\,\mathrm{Tdet}[\Phi(\theta_i-\theta_j\,;\,t)]_{i,j=1,\ldots,N} \tag{28}$$

where

$$\Phi(\theta;\,t)=\begin{pmatrix} S_N(\theta) & D_N(\theta;\,t)\\ J_N(\theta;\,t) & S_N(\theta)\end{pmatrix} \tag{29}$$

with S_N, D_N, J_N given by

$$S_N(\theta)=\frac{1}{2\pi}\sum_{k=-(N-1)/2}^{(N-1)/2}\exp(\mathrm{i}k\theta)=\frac{\sin(N\theta/2)}{2\pi\sin(\theta/2)} \tag{30}$$

$$D_N(\theta;\,t)=-\frac{1}{\pi}\sum_{k>0}^{(N-1)/2}\exp(2k^2t)k\sin(k\theta) \tag{31}$$

$$J_N(\theta;\,t)=-\frac{1}{\pi}\sum_{k=(N+1)/2}^{\infty}\exp(-2k^2t)\,\frac{\sin(k\theta)}{k}. \tag{32}$$

The summation index k acquires, as in (18), (26), integral or half-integral values as N is odd or even.

It is shown in appendix 3 that

$$\int_0^{2\pi}\Phi(\theta-\phi;\,t)\Phi(\phi;\,t)\,\mathrm{d}\phi=\Phi(\theta;\,t)+\tau\Phi(\theta;\,t)-\Phi(\theta;\,t)\tau \tag{33}$$

in which τ is a constant quaternion. Moreover

$$\int_0^{2\pi}\Phi(0;\,t)\,\mathrm{d}\theta=2\pi\Phi(0;\,t)=N. \tag{34}$$

† The expression given in (3.9) is for a_{ji} rather than a_{ij} as stated. The corresponding error in (3.27) does not affect the later results in [23]. Moreover the same result for odd N gives additional terms in (26) which, as shown in appendix 1, do not affect Pf[F] in (25).

Thus theorem 4.1 of [23] can be applied repeatedly on (28) to obtain the correlation functions (20). We get

$$R_n(\theta_1, \ldots, \theta_n; t) = \mathrm{Tdet}[\Phi(\theta_i - \theta_j; t)]_{i,j=1,\ldots,n}. \tag{35}$$

For $t = 0, \infty$, (35) agrees with the results given by Dyson [25] for COE and CUE. For $t \to \infty$, $J_N \to 0$, $D_N \to \infty$ but $J_N D_N \to 0$, so that J_N and D_N can be effectively dropped from (29) since, in the Tdet expansion, they appear only in the combination $J_N D_N$. For the same reason, the negative signs in (31), (32) can also be dropped, but have been retained to make correspondence with the results in [25] as well as the $N \to \infty$ results of [22, 23].

3.3. *Correlation functions for $N \to \infty$*

Since $R_1 = (N/2\pi)$, we write $\theta = 2\pi r/N$. Then, for large N, $S_N(\theta) \to (N/2\pi)S(r)$, $D_N(\theta; t) \to (N/2\pi)^2 D(r; \Lambda)$, $J_N(\theta; t) \to J(r; \Lambda)$ where

$$S(r) = \frac{\sin \pi r}{\pi r} \tag{36}$$

$$D(r; \Lambda) = -\frac{1}{\pi} \int_0^{\pi} \mathrm{d}k \exp(2\Lambda k^2) k \sin(kr) \tag{37}$$

$$J(r; \Lambda) = -\frac{1}{\pi} \int_{\pi}^{\infty} \exp(-2\Lambda k^2) \frac{\sin(kr)}{k} \tag{38}$$

and, as defined in section 1,

$$\Lambda = \left(\frac{N}{2\pi}\right)^2 t. \tag{39}$$

Since J_N, D_N always appear in the combination $J_N D_N$, we can replace $\Phi(\theta; t)$ in (35) by $(N/2\pi)\sigma(r; \Lambda)$ where

$$\sigma(r; \Lambda) = \begin{pmatrix} S(r) & D(r; \Lambda) \\ J(r; \Lambda) & S(r) \end{pmatrix}. \tag{40}$$

Taking $N \to \infty$ and $t \to 0$ limits with Λ finite, we obtain from (22), (35)

$$R_n(r_1, \ldots, r_n; \Lambda) = \mathrm{Tdet}[\sigma(r_i - r_j; \Lambda)]_{i,j=1,\ldots,n} \tag{41}$$

agreeing with the GOE $\to$ GUE results, (2.33)-(2.36) of [23]. Since the equilibrium result is obtained for $\Lambda \to \infty$, the transition takes place for $t = 0(N^{-2})$.

The cluster functions and their Fourier transforms are given in appendix 4.

4. CSE $\to$ CUE ensembles

To obtain the CSE $\to$ CUE results, it is tempting to use the initial distribution $P(\phi; 0) = C_4(N)(Q_N(\phi))^4$, where $C_4(N) = (2\pi)^{-N} 2^{N(2N-1)}((2N)!)^{-1}$. However the CSE is realized in physical systems with doubly-degenerate eigenvalues [28]; it is due to the well known Kramers' degeneracy. The degeneracy is broken for $t \neq 0$. To take proper account of this, we shall consider (14), (19) for $2N$-dimensional matrices with the initial distribution

$$P(\phi_1, \ldots, \phi_{2N}; 0) = C_4(N)(Q_N(\phi))^4 \delta(\phi_1 - \phi_{N+1})\delta(\phi_2 - \phi_{N+2}) \ldots \delta(\phi_N - \phi_{2N}). \tag{42}$$

Similar considerations apply for GSE → GUE ensembles also [23]. The rest of the calculations run parallel to that in section 3.

4.1. Joint-probability density for the eigenangles

To evaluate the integral in (14), we first integrate over the variables $\phi_{N+1}, \ldots, \phi_{2N}$. Because of the δ-functions, the result is the limit $(\phi_{j+N} \to \phi_j)$ of the integrand which is of the form 0/0. To evaluate the limit, note that

$$\frac{(Q_N(\phi_1 \ldots \phi_N))^4}{Q_{2N}(\phi_1 \ldots \phi_{2N})} = \frac{(-1)^{N(N-1)/2} 2^N}{(\phi_1 - \phi_{N+1}) \ldots (\phi_N - \phi_{2N})} + \ldots \tag{43}$$

and

$$\lim (-1)^{N(N-1)/2} \frac{\det[f(\theta_i - \phi_j)]_{i,j=1,\ldots,2N}}{(\phi_1 - \phi_{N+1}) \ldots (\phi_N - \phi_{2N})} = \det\left[f(\theta_i - \phi_j), -\frac{\partial}{\partial \phi_j} f(\theta_i - \phi_j) \right]_{\substack{i=1,\ldots,2N \\ j=1,\ldots,N}}. \tag{44}$$

The other terms on the RHS of (43) do not contribute to the limit. Thus

$$P(\theta; t) = \frac{2^N C_4(N)}{(2N)!} \exp\left(\frac{tN(4N^2-1)}{6} \right) Q_{2N}(\theta) \times \int_0^{2\pi} \ldots \int d\phi_1 \ldots d\phi_N \det\left[f(\theta_i - \phi_j), -\frac{\partial}{\partial \phi_j} f(\theta_i - \phi_j) \right]_{\substack{i=1,\ldots,2N \\ j=1,\ldots,N}}. \tag{45}$$

Using (3.4), (3.7) and (3.11) of [23] we can now integrate over $\phi_1, \ldots, \phi_N$. The result is (see appendix 1)

$$P(\theta; t) = \frac{C_4(N)}{(2N-1)!!} \exp\left(\frac{tN(4N^2-1)}{6} \right) Q_{2N}(\theta) \, \mathrm{Pf}[F_{ij}]_{i,j=1,\ldots,2N} \tag{46}$$

where the $2N$-dimensional antisymmetric matrix $\boldsymbol{F}$ is given by

$$\boldsymbol{F}_{ij} \equiv \boldsymbol{F}(\theta_i - \theta_j; t) = \frac{2}{\pi} \sum_{k=1/2}^{\infty} \exp(-2k^2 t) k \sin k(\theta_i - \theta_j) \tag{47}$$

with k half-integral.

4.2. Correlation functions for finite N

As in section 3.2, we write (46) in the form (appendix 2 gives the proof)

$$P(\theta; t) = (2N!)^{-1} \, \mathrm{Tdet}[\boldsymbol{\Phi}(\theta_i - \theta_j; t)]_{i,j=1,\ldots,2N} \tag{48}$$

where

$$\boldsymbol{\Phi}(\theta; t) = \begin{pmatrix} S_{2N}(\theta) & K_{2N}(\theta; t) \\ I_{2N}(\theta; t) & S_{2N}(\theta) \end{pmatrix} \tag{49}$$

with S_{2N} determined by (30) and I_{2N}, K_{2N} by

$$I_{2N}(\theta; t) = -\frac{1}{\pi} \sum_{k=1/2}^{N-1/2} \exp(2k^2 t) \frac{\sin(k\theta)}{k} \tag{50}$$

$$K_{2N}(\theta; t) = -\frac{1}{\pi} \sum_{k=N+1/2}^{\infty} \exp(-2k^2 t) k \sin(k\theta). \tag{51}$$

Here the summation index k is always half-integral. As shown in appendix 3, Φ also satisfies the relation (33). Moreover the integral for $\Phi(0, t)$ in (34) yields $2N$. Thus making repeated use of theorem (4.1) of [23], we find the correlation functions

$$R_n(\theta_1, \ldots, \theta_n; t) = \mathrm{Tdet}[\Phi(\theta_i - \theta_j; t)]_{i,j=1,\ldots,n}. \tag{52}$$

As in (35), the $t=\infty$ result of (52) agrees with the CUE result of Dyson [25], while the $t=0$ result yields Dyson's CSE result when the double degeneracy is properly incorporated in the latter.

4.3. *Correlation functions for $N \to \infty$*

Since $R_1 = N/\pi$ in this case, we write $\theta = \pi r/N$. For large N, $S_{2N}(\theta) \to (N/\pi)S(r)$, $I_{2N}(\theta; t) \to I(r; \Lambda)$, $K_{2N}(\theta; t) \to (N/\pi)^2 K(r; \Lambda)$, where $S(r)$ is given by (36), I, K are given by

$$I(r; \Lambda) = -\frac{1}{\pi}\int_0^\pi \mathrm{d}k \exp(2\Lambda k^2)\frac{\sin(kr)}{k} \tag{53}$$

$$K(r; \Lambda) = -\frac{1}{\pi}\int_\pi^\infty \mathrm{d}k \exp(-2\Lambda k^2) k \sin(kr) \tag{54}$$

and Λ is

$$\Lambda = \left(\frac{N}{\pi}\right)^2 t. \tag{55}$$

Then, as in section 3.3,

$$\boldsymbol{R}_n(r_1, \ldots, r_n; \Lambda) = \mathrm{Tdet}[\boldsymbol{\sigma}(r_i - r_j; \Lambda)]_{i,j=1,\ldots,n} \tag{56}$$

where

$$\boldsymbol{\sigma}(r; \Lambda) = \begin{pmatrix} S(r) & K(r; \Lambda) \\ I(r; \Lambda) & S(r) \end{pmatrix} \tag{57}$$

agreeing with the GSE $\to$ GUE results, (2.66)–(2.68) of [23]. Again, the transition takes place for $t = \mathrm{O}(N^{-2})$.

See appendix 4 for the cluster functions and their Fourier transforms.

5. Hierarchical relations among correlation functions

The correlation functions $\boldsymbol{R}_n$ are known in closed forms only for the above two cases. There is, however, interest in other initial conditions and other β values (e.g. the Poisson $\to$ COE transition). We ask therefore whether one can obtain information about the large-N forms directly from (9), without explicitly solving it for finite N. A similar question for the $\mathrm{GE}_\beta(t)$ has led [16] to BBGKY-like hierarchic set of relations among the $\boldsymbol{R}_n$ ($\boldsymbol{R}_n$ related to $\boldsymbol{R}_{n+1}$). We shall in this section obtain identical relations for $\mathrm{CE}_\beta(t)$ also; this will, *inter alia*, confirm the validity of the transition parameter Λ for all ensembles.

In section 2 we chose $v^2=1$ so that the results for finite N in sections 3 and 4 can be written compactly. In the present section it will be helpful to keep it general. Thus, in (9), we replace $(\partial P/\partial t)$ by $(1/v^2)(\partial P/\partial t)$. We now integrate both sides of the equation over $\theta_{n+1},\ldots,\theta_N$ and multiply by $N!/(N-n)!$. Then, using (20), we obtain

$$\begin{aligned}\frac{1}{v^2}\frac{\partial R_n}{\partial t}&=\sum_{j=1}^{n}\frac{\partial^2 R_n}{\partial\theta_j^2}-\tfrac{1}{2}\beta\sum_{i\neq j}^{n}\frac{\partial}{\partial\theta_j}R_n\cot\left(\frac{\theta_j-\theta_i}{2}\right)\\&\quad-\tfrac{1}{2}\beta\sum_{j=1}^{n}\frac{\partial}{\partial\theta_j}\int_0^{2\pi}R_{n+1}\cot\left(\frac{\theta_j-\theta_{n+1}}{2}\right)\mathrm{d}\theta_{n+1}\\&=\sum_{j=1}^{n}\frac{\partial}{\partial\theta_j}|Q_n|^\beta\frac{\partial}{\partial\theta_j}\frac{R_n}{|Q_n|^\beta}-\tfrac{1}{2}\beta\sum_{j=1}^{n}\frac{\partial}{\partial\theta_j}\int_0^{2\pi}R_{n+1}\cot\left(\frac{\theta_t-\theta_{n+1}}{2}\right)\mathrm{d}\theta_{n+1}.\end{aligned}\tag{58}$$

Here we have made use of the fact that the integral

$$\frac{N!}{(N-n)!}\int_0^{2\pi}\cdots\int\mathrm{d}\theta_{n+1}\ldots\mathrm{d}\theta_N\frac{\partial^2P}{\partial\theta_j^2}$$

is zero for $j>n$ and is $(\partial^2R_n/\partial\theta_j^2)$ for $1\leqslant j\leqslant n$. Similarly the integral

$$\frac{N!}{(N-n)!}\int\cdots\int\mathrm{d}\theta_{n+1}\ldots\mathrm{d}\theta_N\frac{\partial}{\partial\theta_j}P\cot\left(\frac{\theta_j-\theta_i}{2}\right)$$

is zero for $j>n$; for $1\leqslant j\leqslant n$, its value is

$$\frac{\partial}{\partial\theta_j}\left(R_n\cot\left(\frac{\theta_j-\theta_i}{2}\right)\right)$$

if $1\leqslant i\leqslant n$, and is

$$\frac{1}{(N-n)}\frac{\partial}{\partial\theta_j}\int\mathrm{d}\theta_{n+1}R_{n+1}(\theta_1,\ldots,\theta_{n+1})\cot\left(\frac{\theta_j-\theta_{n+1}}{2}\right)$$

if $i>n$. Note that, for $n=N$, $R_{N+1}=0$, so that (58) contains (9) as a special case. We now take the limit $N\to\infty$ for fixed n.

We begin with $R_1(\theta;t)$. This is needed since it fixes the scale for the eigenangle fluctuations. It is convenient to write $R_1=N\bar{\rho}(\theta;t)$ so that $\bar{\rho}$ is normalized to unity, and choose $\beta v^2N=1$; these choices are also encountered in the studies of the Gaussian ensembles [16, 21]. Then (58) yields for large N

$$\frac{\partial\bar{\rho}(\theta)}{\partial t}=-\frac{1}{2}\frac{\partial}{\partial\theta}\left[\bar{\rho}(\theta)\int_0^{2\pi}\mathrm{d}\phi\cot\left(\frac{\theta-\phi}{2}\right)\bar{\rho}(\phi)\right]\tag{59}$$

where a principal-value integral is taken. This nonlinear equation gives uniquely $\bar{\rho}(\theta;t)$ starting from an initial $\bar{\rho}(\theta;0)$. Terms ignored in (59) are $N^{-1}(\partial^2\rho/\partial\theta^2)$ and the integral involving $(N^{-2}R_2(\theta,\phi)-\bar{\rho}(\theta)\bar{\rho}(\phi))$; both are of order N^{-1} or lower. Equation (59) can be written in a more compact form if we introduce the transform

$$\bar{g}(\psi;t)=\tfrac{1}{2}\int_0^{2\pi}\mathrm{d}\theta\cot\left(\frac{\psi-\theta}{2}\right)\bar{\rho}(\theta;t)\tag{60}$$

where the 'angle' ψ is complex. The transform gives back the density uniquely via the relation

$$\bar{\rho}(\theta)=\frac{1}{\pi}\operatorname{Im}\bar{g}(\theta-\mathrm{i}\delta)\tag{61}$$

where δ is a small positive number; (the real part is related to the principal-value integral in (59)). Then (59) is equivalent to

$$\frac{\partial\bar{g}}{\partial t}=\frac{\partial\bar{g}^2}{\partial\psi} \tag{62}$$

and hence to

$$\bar{g}(\psi;t)=\bar{g}(\psi-t\bar{g}(\psi;t);0). \tag{63}$$

Equations (59), (63) may be called the Dyson–Pastur equations, being analogous to their results for $\mathrm{GE}_\beta(t)$; see [16, 20, 21, 32]. It can be shown that, for large t, $\bar{\rho}(\theta)=(2\pi)^{-1}$. Similarly, if $\bar{\rho}(\theta;0)=(2\pi)^{-1}$, then $\bar{\rho}(\theta;t)=(2\pi)^{-1}$ for all t. Similar results are obtained in the Gaussian cases also where, instead of the uniform density, one uses the (equilibrium) semicircular density. However, unlike the Gaussian cases, $\bar{\rho}(\theta;0)=\delta(\theta)$ does not lead to the equilibrium density for all $t>0$.

For $n>1$, we consider eigenangles in the neighbourhood of θ and write as in (21), (22), $\theta_j=\theta+r_j/N\bar{\rho}$ and $R_n=\boldsymbol{R}_n(N\bar{\rho})^n$ where $N\bar{\rho}=R_1(\theta)$. Then, for large N, the last form of (58) is $\mathrm{O}(N^{n+2})$ while, with fixed tv^2, the first form is $(1/v^2)(\partial R_n/\partial t)=\mathrm{O}(N^n)$. This implies that the transition takes place for finite values of tv^2N^2 (and hence small tv^2). We therefore define, as in section 1, $\Lambda=tv^2N^2\bar{\rho}^2$. Let us agree that, for $t=0$, $\bar{\rho}$ is not singular (nor zero) in the neighbourhood of θ and the $\boldsymbol{R}_n$ are also well defined. Then (59) implies that, for finite Λ, $\bar{\rho}$ varies little from its '$t=0$' value. Hence, keeping only $\mathrm{O}(N^{n+2})$ terms,

$$\frac{1}{v^2}\frac{\partial R_n}{\partial t}=(N\bar{\rho})^{n+2}\frac{\partial\boldsymbol{R}_n}{\partial\Lambda}. \tag{64}$$

Similarly,

$$\sum_{j=1}^{n}\frac{\partial}{\partial\theta_j}|Q_N|^\beta\frac{\partial}{\partial\theta_j}\frac{R_n}{|Q_N|^\beta}=(N\bar{\rho})^{n+2}\sum_{j=1}^{n}\frac{\partial}{\partial r_j}|\Delta_n|^\beta\frac{\partial}{\partial r_j}\frac{\boldsymbol{R}_n}{|\Delta_n|^\beta}. \tag{65}$$

Here $\Delta_n=\Delta_n(r_1,\ldots,r_n)$ of (23); note that for small separations $Q_N(\theta)\simeq\Delta_n(\theta)$; also terms involving $(\partial\bar{\rho}/\partial\theta)$ will be of lower order. To evaluate the integral term in (58) we divide the range of integration of θ_{n+1} into two parts. In the first part, $|\theta_j-\theta_{n+1}|=\mathrm{O}(N^{-1})$ and hence $R_{n+1}\to(N\bar{\rho})^{n+1}\boldsymbol{R}_{n+1}$ and $\mathrm{d}\theta_{n+1}/(2\cot((\theta_j-\theta_{n+1})/2))\to\mathrm{d}r_{n+1}/(r_j-r_{n+1})$ with the range of integration being $(-\infty,\infty)$. Ignoring $\partial\bar{\rho}/\partial\theta$, this part contributes

$$-(N\bar{\rho})^{n+2}\beta\sum_{j=1}^{n}\frac{\partial}{\partial r_j}\int_{-\infty}^{\infty}\boldsymbol{R}_{n+1}\frac{\mathrm{d}r_{n+1}}{(r_j-r_{n+1})}. \tag{66}$$

In the second part, θ_{n+1} is uncorrelated with θ_j and hence $R_{n+1}\to R_n\cdot R_1(\theta_{n+1})$; this gives, to $\mathrm{O}(N^{n+2})$,

$$-\frac{\beta}{2}N^{n+2}\bar{\rho}^{n+1}\left(\int_0^{2\pi}\bar{\rho}(\phi)\cot\left(\frac{\theta-\phi}{2}\right)\mathrm{d}\phi\right)\left(\sum_{j=1}^{n}\frac{\partial}{\partial r_j}\boldsymbol{R}_n\right)=0 \tag{67}$$

since $\Sigma\,(\partial/\partial r_j)\boldsymbol{R}_n=0$. Substituting (64)–(67) in (58), we find the hierarchical relations for $N\to\infty$:

$$\frac{\partial\boldsymbol{R}_n}{\partial\Lambda}=\sum_{j=1}^{n}\frac{\partial}{\partial r_j}|\Delta_n|^\beta\frac{\partial}{\partial r_j}\frac{\boldsymbol{R}_n}{|\Delta_n|^\beta}-\beta\sum_{j=1}^{n}\frac{\partial}{\partial r_j}\int_{-\infty}^{\infty}\boldsymbol{R}_{n+1}\frac{\mathrm{d}r_{n+1}}{(r_j-r_{n+1})} \tag{68}$$

which is the same as (71) of [16] for the GE_β.

We emphasize that, for finite v^2, the approach to equilibrium is very rapid for the density $\bar{\rho}$ as well as the correlations $\boldsymbol{R}_n$. However, for $v^2N=1$, the former transition will be smooth while the latter is still discontinuous. It is the rapid transition in $\boldsymbol{R}_n$ which should be observed in the spectra of time-evolution operators of chaotic systems with time-periodic Hamiltonians. We also mention here that the general solution of (68) is still unknown, but some approximate solutions (for cases other than those in sections 3 and 4) can be worked out [16].

6. Conclusion

The equilibrium ensembles (Gaussian as well as circular) derive from a minimum information principle [33] in which, apart from the exact symmetries, no other information about the system is taken into account. The Brownian ensembles provide a natural framework, consistent with the principle, for studying the gradual breaking of one or more symmetries. We find that, just as in the Gaussian cases, the eigenvalue correlations in the circular ensembles display an extreme sensitivity with respect to small symmetry breaking; equilibrium is reached rapidly. For large matrices, the two ensembles yield identical eigenvalue correlations under similar conditions. The ensembles are moreover ergodic [34]; almost all members of the ensemble display the same eigenvalue correlations as the ensemble-averaged ones. We conjecture that the evolution operators of quantum maps with chaotic classical limits are characteristic members of the circular type Brownian ensembles.

The two-eigenvalue correlation function $\boldsymbol{R}_2$ usually gives the important observable correlation effects. Specifically, the small-frequency behaviour of the Fourier transform $(1-b_2(k)$; see appendix 4) of $(\delta(r)-1+\boldsymbol{R}_2(r))$ (where $\delta(r)$ is included to take account of the self-correlation) determines whether an eigenvalue spectrum is rigid or uncorrelated; a suppression of the amplitudes for small frequencies implies rigidity. Berry [6] has shown that the small-frequency behaviour in autonomous chaotic systems is like that in the equilibrium ensembles; the proofs rely on a semiclassical quantization of chaotic systems involving the classical periodic orbits [35], and a principle of uniformity for the distribution of periodic orbits and their amplitudes [36]. Extending these ideas to quantum maps [37] and using the methods of [8] for small TR-non-invariance, we have recently [38] proved the following result: the small-$|k|$ form of $(1-b_2(k))$ of a chaotic map with small TR-non-invariance is given exactly by the corresponding result (109) of appendix 4 for the COE $\to$ CUE and CSE $\to$ CUE transitions. An expression for Λ is also obtained which, by the methods of [39], is independently identified as the local symmetry-breaking parameter. This gives a confirmation of our above conjecture.

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Appendix 1. Proof of (26), (27), (47)

We start with the proof of (26) for even N. Using the corrected form of (3.9) of [23] we have

$$F_{ij} = \int_{2\pi \geqslant \phi_1 \geqslant \phi_2 \geqslant 0} \int d\phi_1\, d\phi_2 [f(\theta_i - \phi_1) f(\theta_j - \phi_2) - f(\theta_i - \phi_2) f(\theta_j - \phi_1)]$$

$$= \frac{1}{4\pi^2} \sum_{k_1, k_2 = -\infty}^{\infty} \exp(-(k_1^2 + k_2^2)t) \exp(i(k_1\theta_i + k_2\theta_j))$$

$$\times \iint_{\phi_1 \geqslant \phi_2} d\phi_1\, d\phi_2 [\exp(-(ik_1\phi_1 + ik_2\phi_2)) - \exp(-(ik_2\phi_1 + ik_1\phi_2))]$$

$$= \frac{1}{4\pi^2} \sum_{k_1, k_2 = -\infty}^{\infty} \exp(-(k_1^2 + k_2^2)t) \exp(i(k_1\theta_i + k_2\theta_j)) \left(-\frac{4\pi i}{k_1} \delta_{k_1, -k_2} \right)$$

$$= \frac{1}{i\pi} \sum_{k=-\infty}^{\infty} \exp(-2k^2 t) \exp(ik(\theta_i - \theta_j))/k \tag{69}$$

which yields (26). Here in the second step we have used (18) and in the third step evaluated the double integral; the summation indices are all half-integral.

For odd N, we first write (25) with F replaced by an $(N+1)$-dimensional anti-symmetric matrix $\hat{F}$. Then, from (3.9), (3.10) of [23], with $i, j = 1, \ldots, N$,

$$\hat{F}_{N+1, N+1} = 0 \tag{70}$$

$$\hat{F}_{i, N+1} = -\hat{F}_{N+1, i} = \int_0^{2\pi} d\phi\, f(\phi) = 1 \tag{71}$$

and

$$\hat{F}_{ij} = \iint_{2\pi \geqslant \phi_1 \geqslant \phi_2 \geqslant 0} d\phi_1\, d\phi_2 [f(\theta_i - \phi_1) f(\theta_j - \phi_2) - f(\theta_i - \phi_2) f(\theta_j - \phi_1)]$$

$$= F_{ij} - a_i + a_j. \tag{72}$$

Here in (71), (72) the summation indices are all integral. The integral in (71) is then straightforward; only the $k = 0$ term gives non-zero contribution. For the integral in (72) we follow the similar steps in (69), but now there are additional terms (which give a_i and a_j in (72)) coming from $(k_1 \neq 0, k_2 = 0)$ and $(k_1 = 0, k_2 \neq 0)$; moreover, in F_{ij}, $k_1 = k_2 = 0$ should be excluded. Then F_{ij} is given by (26) and a_i by

$$a_i = \frac{2}{\pi} \sum_{k=1}^{\infty} \exp(-2k^2 t) \sin(k\theta_i)/k. \tag{73}$$

Finally (25)–(27) are recovered in the stated form, if $\det F = \det \hat{F}$ (so that $\mathrm{Pf}(\hat{F}) = \mathrm{Pf}(F)$). To prove the latter, we do the following operations in $\det \hat{F}$ sequentially: $R_N - R_{N-1}$, $R_{N-1} - R_{N-2}, \ldots, R_2 - R_1$, $C_N - C_{N-1}$, $C_{N-1} - C_{N-2}$, $C_2 - C_1$, where R_i and C_j stand for the ith row and jth column respectively. After these operations we expand the determinant about the last column and then about the last row. The resultant $(N-1)$-dimensional determinant is independent of the a_i; hence the proof.

To prove (47) we use (3.11) of [23]. We have, with half-integral summation indices,

$$F_{ij}=\int_0^{2\pi} d\phi[f(\theta_i-\phi)\left(-\frac{\partial f(\theta_j-\phi)}{\partial\phi}\right)-f(\theta_j-\phi)\left(-\frac{\partial f(\theta_i-\phi)}{\partial\phi}\right)]$$

$$=\frac{1}{4\pi^2}\sum_{k_1,k_2=-\infty}^{\infty}\exp(-(k_1^2+k_2^2)t)\exp(i(k_1\theta_i+k_2\theta_j))(-4\pi i k_1\delta_{k_1,-k_2})$$

$$=\frac{1}{i\pi}\sum_{k=-\infty}^{\infty}k\exp(-2k^2t)\exp(ik(\theta_i-\theta_j)) \tag{74}$$

which yields (47).

Appendix 2. Proof of (28), (48)

Using the methods of [23, 25], we verify here that the joint-probability density of eigenangles in the COE→CUE and CSE→CUE ensembles can be expressed as a Tdet of a self-dual quaternion matrix.

We start with (28). We define $g(\theta_i-\theta_j)=F_{ij}/2$. Then

$$g(\theta)+J_N(\theta)=\frac{1}{\pi}\sum_{k>0}^{(N-1)/2}\exp(-2k^2t)\sin(k\theta)/k. \tag{75}$$

It would be good to write $\sin k\theta=(e^{ik\theta}-e^{-ik\theta})/2i$ in (30)-(32), (75) in the following calculations. Consider now even N, in which case the index k is half-integral. The $2N$-dimensional matrix product,

$$G=\begin{pmatrix}\exp(ik\theta_i) & 0\\ (ik)^{-1}\exp(-2k^2t+ik\theta_i) & 0\end{pmatrix}\begin{pmatrix}\exp(-ik\theta_j) & ik\exp(2k^2t-ik\theta_j)\\ 0 & 0\end{pmatrix}$$

$$=\begin{pmatrix}S_N(\theta_i-\theta_j) & D_N(\theta_i-\theta_j)\\ g(\theta_i-\theta_j)+J_N(\theta_i-\theta_j) & S_N(\theta_i-\theta_j)\end{pmatrix} \tag{76}$$

has rank N, so that its rows $(g+J_N, S_N)$ are linear combinations of the rows (S_N, D_N). Therefore $\det[\Phi]$ is not changed when we subtract the rows (g_N+J_N, S_N) from the rows (J_N, S_N) of Φ. The subtraction gives

$$\det[\Phi]=\begin{pmatrix}S_N & D_N\\ -g & 0\end{pmatrix}=\det[g]\det[D_N]. \tag{77}$$

Now,

$$\det[g]=2^{-N}\det[F]=2^{-N}(\mathrm{Pf}[F])^2 \tag{78}$$

and

$$\det[D_N]=(2\pi)^{-N}\det[\exp(-ik\theta_j)]\det[ik\exp(2k^2t+ik\theta_i)]$$

$$=(2\pi)^{-N}\prod_{k=-(N-1)/2}^{(N-1)/2}(ik\exp(2k^2t))|\det[\exp(ik\theta_j)]|^2$$

$$=2^N(N!)^2(C_1(N))^2\exp[N(N^2-1)t/6](Q_N(\theta))^2 \tag{79}$$

so that, from (25), (77)-(79),

$$\det[\Phi]=(N!P(\theta;t))^2 \tag{80}$$

yielding (28).

For odd N, the index k is integral and therefore $k=0$ is also included. We then define the $2N$-dimensional matrix product G_δ obtained from G by the replacements,

$$\begin{aligned}&(\mathrm{i}k)^{-1}\exp(-2k^2t+\mathrm{i}k\theta_i)\to\delta^{-1}\\&\mathrm{i}k\exp(2k^2t-\mathrm{i}k\theta_j)\to\delta\end{aligned}\tag{81}$$

for the elements with $k=0$, where δ is arbitrary. Then

$$G_\delta=\begin{pmatrix}S_N(\theta_i-\theta_j) & D_N(\theta_i-\theta_j)+(\delta/2\pi)\\ g(\theta_i-\theta_j)+J_N(\theta_i-\theta_j)+(2\pi\delta)^{-1} & S_N(\theta_i-\theta_j)\end{pmatrix}\tag{82}$$

which is still of rank N. Instead of $[\Phi(\theta_i-\theta_j)]$, we consider the matrix

$$[\Phi_\delta]=\begin{pmatrix}S_N(\theta_i-\theta_j) & D_N(\theta_i-\theta_j)+(\delta/2\pi)\\ J_N(\theta_i-\theta_j) & S_N(\theta_i-\theta_j)\end{pmatrix}.\tag{83}$$

The determinant of $[\Phi_\delta]$ is unchanged by subtracting from the rows $[J_N, S_N]$ the corresponding rows of G_δ. Thus

$$\begin{aligned}\det[\Phi_\delta]&=\begin{pmatrix}S_N & D_N+(\delta/2\pi)\\ -g-(2\pi\delta)^{-1} & 0\end{pmatrix}\\&=\det[g+(2\pi\delta)^{-1}]\det[D_N+(\delta/2\pi)].\end{aligned}\tag{84}$$

Now,

$$\begin{aligned}\det[g+(2\pi\delta)^{-1}]&=\begin{pmatrix}g(\theta_i-\theta_j) & 0 & 1\\ 0 & (2\pi\delta)^{-1} & 1\\ -1 & -1 & 0\end{pmatrix}\\&=(2\pi\delta)^{-1}\begin{pmatrix}g(\theta_i-\theta_j) & 1\\ -1 & 0\end{pmatrix}+\begin{pmatrix}g(\theta_i-\theta_j) & 0 & 1\\ 0 & 0 & 1\\ -1 & -1 & 0\end{pmatrix}\\&=(2^N\pi\delta)^{-1}\det[F]\end{aligned}\tag{85}$$

where, in the second step, the second determinant is zero, being that of an antisymmetric matrix of odd dimension $(N+2)$. $\mathrm{Det}[D_N+(\delta/2\pi)]$ can be evaluated as in (79) by using the second replacement of (81) for $k=0$. We find then

$$\det[D_N+(\delta/2\pi)]=(2^N\pi\delta)(N!\,C_1(N)Q_N(\theta))^2\exp[N(N^2-1)t/6].\tag{86}$$

Equations (25), (83)-(86) yield (80), and hence (28), for $\delta\to0$.

We now turn to (48). We define $h(\theta_i-\theta_j)=F_{ij}/2$ so that

$$h(\theta)+K_{2N}(\theta)=\frac{1}{\pi}\sum_{k=1/2}^{N-1/2}k\exp[-2k^2t]\sin(k\theta).\tag{87}$$

The $4N$-dimensional matrix product,

$$\begin{aligned}G&=(2\pi)^{-1}\begin{pmatrix}\exp(\mathrm{i}k\theta_i) & 0\\ (-\mathrm{i}k)^{-1}\exp[2k^2t+\mathrm{i}k\theta_i] & 0\end{pmatrix}\\&\quad\times\begin{pmatrix}\exp(-\mathrm{i}k\theta_j) & -\mathrm{i}k\exp[-2k^2t-\mathrm{i}k\theta_j]\\ 0 & 0\end{pmatrix}\\&=\begin{pmatrix}S_{2N}(\theta_i-\theta_j) & h(\theta_i-\theta_j)+K_{2N}(\theta_i-\theta_j)\\ I_{2N}(\theta_i-\theta_j) & S_{2N}(\theta_i-\theta_j)\end{pmatrix}\end{aligned}\tag{88}$$

is of rank $2N$. Hence $\det[\Phi]$ is unchanged, if we subtract from its columns $\begin{pmatrix} K_{2N} \\ S_{2N} \end{pmatrix}$ the corresponding columns of G. This gives

$$\det[\Phi]=\det\begin{pmatrix} S_{2N} & -h \\ I_{2N} & 0 \end{pmatrix}=\det[h]\det[I_{2N}]. \tag{89}$$

Then, as in (78), (79) above, we have

$$\det[h]=2^{-2N}(\mathrm{Pf}[F])^2 \tag{90}$$

and

$$\det[I_{2N}]=2^{2N}((2N)!\,C_4(N)Q_{2N}(\theta))^2\exp[N(4N^2-1)t/3] \tag{91}$$

so that, from (46)

$$\det[\Phi]=((2N)!\,P(\theta;t))^2 \tag{92}$$

yielding (48).

Appendix 3. Verification of (33)

For any two functions $f_1(\theta)$, $f_2(\theta)$, we define the composition

$$f_1*f_2=\int_0^{2\pi} f_1(\theta-\phi)f_2(\phi)\,\mathrm{d}\phi. \tag{93}$$

Then, for the COE → CUE ensembles, the definitions (30)–(32)

$$S_N*S_N=S_N \tag{94}$$

$$D_N*S_N=S_N*D_N=D_N \tag{95}$$

$$J_N*S_N=S_N*J_N=0 \tag{96}$$

$$J_N*D_N=D_N*J_N=0 \tag{97}$$

which along with (29) yield (33) with

$$\tau=\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{98}$$

Similarly, for the CSE → CUE ensembles, we have, from (30), (50), (51),

$$S_{2N}*S_{2N}=S_{2N} \tag{99}$$

$$I_{2N}*S_{2N}=S_{2N}*I_{2N}=I_{2N} \tag{100}$$

$$K_{2N}*S_{2N}=S_{2N}*K_{2N}=0 \tag{101}$$

$$K_{2N}*D_{2N}=D_{2N}*K_{2N}=0 \tag{102}$$

which, for $\Phi*\Phi$, yield an equation of the type (33) with

$$\boldsymbol{\tau}=\frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{103}$$

Appendix 4. Cluster functions and Fourier transforms

The n-angle cluster functions Y_n are essentially the n-angle correlation functions $\boldsymbol{R}_n$ from which the lower-order correlation effects have been subtracted out [23, 25]. For example, for the unfolded spectra, $Y_1 = \boldsymbol{R}_1 = 1$, $Y_2 = 1 - \boldsymbol{R}_2$ etc. For the COE→CUE transitions, these are given by

$$Y_n(r_1, \ldots, r_n; \Lambda) = \sum_p \sigma(r_1 - r_2)\sigma(r_2 - r_3), \ldots, \sigma(r_n - r_1) \tag{104}$$

where Σ_p denotes a sum over the $(n-1)!$ distinct cyclic permutations of the indices $(1, 2, \ldots, n)$ and σ is defined in (40). Like the $\boldsymbol{R}_n$ in (41), Y_n is also a scalar and its scalar nature can be made explicit by inserting the operation $\frac{1}{2}$tr before the product in (104). Its Fourier transform is given by

$$\begin{aligned}\delta(k_1 + \ldots + k_n) b_n(k_1, \ldots, k_n; \Lambda) &\equiv \int_{-\infty}^{\infty} \ldots \int \mathrm{d}r_1 \ldots \mathrm{d}r_n \, Y_n(r_1 \ldots r_n; \Lambda) \exp\left(2\pi \mathrm{i} \sum_{j=1}^{n} r_j k_j\right) \\ &= \int_{-\infty}^{\infty} \mathrm{d}p \sum \theta(p)\theta(p + k_1) \ldots \theta(p + k_1 + \ldots + k_{n-1})\end{aligned} \tag{105}$$

where $\theta(k)$ is the Fourier transform of $\sigma(r)$. Dropping some factors which do not affect the value of b_n, we have

$$\theta(k) = \begin{pmatrix} \varepsilon(k) & k \exp(8\pi^2 k^2 \Lambda)\varepsilon(k) \\ -k^{-1} \exp(-8\pi^2 k^2 \Lambda)(1 - \varepsilon(k)) & \varepsilon(k) \end{pmatrix} \tag{106}$$

with

$$\varepsilon(k) = \begin{cases} 1 & |k| < \frac{1}{2} \\ 0 & |k| > \frac{1}{2}. \end{cases} \tag{107}$$

For $n = 2$, b_2 is a function of $(k_1 - k_2)$ (since Y_2 is a function of $r_1 - r_2$). Then

$$\begin{aligned} b_2(k) &= 1 - |k| - \int_{(1/2)-|k|}^{1/2} \mathrm{d}p \frac{p}{|k| + p} \exp(-8\pi^2 \Lambda(2p|k| + k^2)) && |k| < 1 \\ &= -\int_{-1/2}^{1/2} \mathrm{d}p \frac{p}{|k| + p} \exp(-8\pi^2 \Lambda(2p|k| + k^2)) && |k| > 1. \end{aligned} \tag{108}$$

For $|k| \ll 1$, this is given by

$$1 - b_2(k) = |k|\{1 + \exp(-8\pi^2 \Lambda |k|)\}. \tag{109}$$

For CSE→CUE, we again have (104) with $\boldsymbol{\sigma}$ given by (57) and hence (105) with θ given by

$$\boldsymbol{\theta}(k) = \begin{pmatrix} \varepsilon(k) & k \exp(-8\pi^2 k^2 \Lambda)(1 - \varepsilon(k)) \\ -k^{-1} \exp(8\pi^2 k^2 \Lambda)\varepsilon(k) & \varepsilon(k) \end{pmatrix} \tag{110}$$

The corresponding b_2 is given by

$$\begin{aligned} b_2(k) &= 1 - |k| - \int_{(1/2)-|k|}^{1/2} \mathrm{d}p \frac{|k| + p}{p} \exp(-8\pi^2 \Lambda(2p|k| + k^2)) && |k| < 1 \\ &= -\int_{-1/2}^{1/2} \mathrm{d}p \frac{|k| + p}{p} \exp(-8\pi^2 \Lambda(2p|k| + k^2)) && |k| > 1. \end{aligned} \tag{111}$$

For $|k| \ll 1$, we again have (109).

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